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Scale invariance rule in electromagnetic scattering

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Abstract

The volume integral equation formalism is used to prove the scale invariance rule for an arbitrarily sized scatterer with an arbitrary shape, morphology, and orientation. The only assumptions are that the scatterer is made of optically isotropic linear materials and is embedded in a homogeneous, linear, isotropic, and nonabsorbing infinite medium.
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1. Introduction

A fundamental property of electromagnetic scattering is the so-called scale invariance rule (SIR). For homogeneous or layered spheres, this rule follows directly from the formulas of the Lorenz–Mie theory. It can also be derived for homogeneous nonspherical scatterers from the T-matrix formulas (see Section 5.8.2 of [1]). However, a general derivation of the SIR for a particle with arbitrary size, shape, morphology, and orientation has not been given yet and is the subject of this short note. In order to avoid redundancy and save space, we will assume that the reader is familiar with [1] and will use exactly the same terminology and notation.

2. Proof of the scale invariance rule

As in Chapter 2 of [1], we consider a finite scattering object in the form of an arbitrary single body or an arbitrary fixed aggregate embedded in an infinite, homogeneous, linear, isotropic, and nonabsorbing medium. The interior of the particle is assumed to be filled with an isotropic, linear, and possibly inhomogeneous material (Fig. 1). We start off where Sections 2.1 and 2.2 of [1] end by introducing the following dimensionless quantities:

$$\check{\mathbf{r}} = k_1 \mathbf{r}, \tag{1}$$

$$\check{\overleftrightarrow{G}}(\check{\mathbf{r}}, \check{\mathbf{r}}') = (\overleftrightarrow{I} + \check{\nabla} \otimes \check{\nabla}) \frac{\exp(\mathrm{i}|\check{\mathbf{r}} - \check{\mathbf{r}}'|)}{4\pi|\check{\mathbf{r}} - \check{\mathbf{r}}'|}, \tag{2}$$

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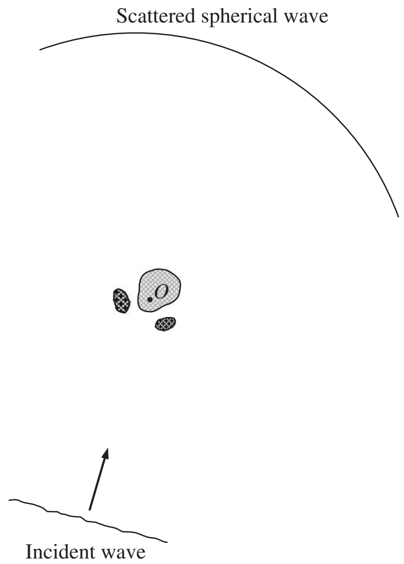

Fig. 1. Schematic representation of the electromagnetic scattering problem. The unshaded exterior region V_{EXT} is unbounded in all directions, whereas the shaded area represents the interior region V_{INT}.

$$\check{\nabla} = \frac{1}{k_1}\nabla, \tag{3}$$

$$\overset{\leftrightarrow}{\check{T}}(\check{\mathbf{r}},\check{\mathbf{r}}') = \frac{1}{k_1^5}\overset{\leftrightarrow}{T}(\mathbf{r},\mathbf{r}'), \tag{4}$$

$$\check{m}(\check{\mathbf{r}}) = m(\mathbf{r}), \tag{5}$$

where $\mathbf{r}$ is the radius vector connecting the origin O of the laboratory reference frame and the observation point, k_1 is the wave number in the host medium, $\overset{\leftrightarrow}{T}(\mathbf{r},\mathbf{r}')$ is the dyadic transition operator defined by Eq. (2.17) of [1], and $m(\mathbf{r})$ is the refractive index of the scatterer relative to that of the host medium. It is then rather obvious that the Lippmann–Schwinger equation (Eq. (2.18) of [1]) can be rewritten in the form

$$\begin{aligned}\overset{\leftrightarrow}{\check{T}}(\check{\mathbf{r}},\check{\mathbf{r}}') &= [\check{m}^2(\check{\mathbf{r}})-1]\delta(\check{\mathbf{r}}-\check{\mathbf{r}}')\overset{\leftrightarrow}{I}\\ &\quad+[\check{m}^2(\check{\mathbf{r}})-1]\int_{\check{V}_{\mathrm{INT}}}\mathrm{d}\check{\mathbf{r}}''\,\overset{\leftrightarrow}{\check{G}}(\check{\mathbf{r}},\check{\mathbf{r}}'')\cdot\overset{\leftrightarrow}{\check{T}}(\check{\mathbf{r}}'',\check{\mathbf{r}}'),\quad \check{\mathbf{r}},\check{\mathbf{r}}'\in\check{V}_{\mathrm{INT}},\end{aligned} \tag{6}$$

where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function and the dimensionless "volume" $\check{V}_{\mathrm{INT}}$ is obtained from the actual volume of the scatterer V_{INT} by multiplying all dimensions of the latter by k_1. It is also taken into account that

$$\delta(\mathbf{r}) = a^3\delta(a\mathbf{r}). \tag{7}$$

Solving Eq. (6) by iteration shows that the dimensionless dyadic transition operator $\overset{\leftrightarrow}{\check{T}}(\check{\mathbf{r}},\check{\mathbf{r}}')$ depends on the dimensionless particle volume $\check{V}_{\mathrm{INT}} = k_1^3 V_{\mathrm{INT}}$ rather than on the actual particle volume and on the wave number separately.

The next steps are to assume that the scattering object is illuminated by a plane electromagnetic wave, to consider scattering in the far-field zone, and to introduce the dimensionless scattering dyadic as follows:

$$\overset{\leftrightarrow}{\check{A}}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}}) = k_1\overset{\leftrightarrow}{A}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}}), \tag{8}$$

where $\overset{\leftrightarrow}{A}(\hat{\mathbf{n}}^{\rm sca}, \hat{\mathbf{n}}^{\rm inc})$ is the scattering dyadic defined by Eq. (2.26) of [1]. Then Eq. (2.27) of [1] takes the form

$$\breve{\overset{\leftrightarrow}{A}}(\hat{\mathbf{n}}^{\rm sca}, \hat{\mathbf{n}}^{\rm inc}) = \frac{1}{4\pi}(\overset{\leftrightarrow}{I} - \hat{\mathbf{n}}^{\rm sca} \otimes \hat{\mathbf{n}}^{\rm sca}) \cdot \int_{\breve{V}_{\rm INT}} {\rm d}\breve{\mathbf{r}}' \exp(-{\rm i}\hat{\mathbf{n}}^{\rm sca} \cdot \breve{\mathbf{r}}') \times \int_{\breve{V}_{\rm INT}} {\rm d}\breve{\mathbf{r}}'' \breve{\overset{\leftrightarrow}{T}}(\breve{\mathbf{r}}', \breve{\mathbf{r}}'') \exp({\rm i}\hat{\mathbf{n}}^{\rm inc} \cdot \breve{\mathbf{r}}''), \tag{9}$$

which shows again that the dimensionless scattering dyadic is a function of the dimensionless particle volume rather than a function of the actual particle volume as well as of the wave number. Of course, the same is true of the dimensionless amplitude scattering matrix $\breve{\mathbf{S}}(\hat{\mathbf{n}}^{\rm sca}, \hat{\mathbf{n}}^{\rm inc}) = k_1 \mathbf{S}(\hat{\mathbf{n}}^{\rm sca}, \hat{\mathbf{n}}^{\rm inc})$, where $\mathbf{S}(\hat{\mathbf{n}}^{\rm sca}, \hat{\mathbf{n}}^{\rm inc})$ is the amplitude scattering matrix defined by Eq. (2.30) of [1].

The SIR is a direct consequence of these results and states the following. If one multiplies all linear dimensions of the scattering object by a constant factor f (thereby not changing the shape and morphology of the object and its orientation with respect to the laboratory coordinate system) and multiplies the wave number k_1 by a factor $1/f$ then the dimensionless scattering dyadic and the dimensionless amplitude scattering matrix of the object do not change.

3. Discussion

The SIR can be reformulated as follows. Consider a class of geometrically similar objects with geometrically similar spatial distributions of the relative refractive index and the same orientation with respect to the laboratory reference frame (Fig. 2). It is clear that each object from the class can be uniquely identified by the value of a typical linear dimension a (for example, the largest or the smallest dimension of the object or the radius of the surface- or volume-equivalent sphere). Then the SIR implies that the dimensionless scattering characteristics of the objects do not depend on specific values of a and k_1, but rather depend on the product of a and k_1 traditionally called the size parameter x.

The size parameter can also be expressed in terms of the wavelength of the incident wave in the exterior region, $\lambda = 2\pi/k_1$, as $x = 2\pi a/\lambda$. This means that multiplying the typical particle size and the wavelength by

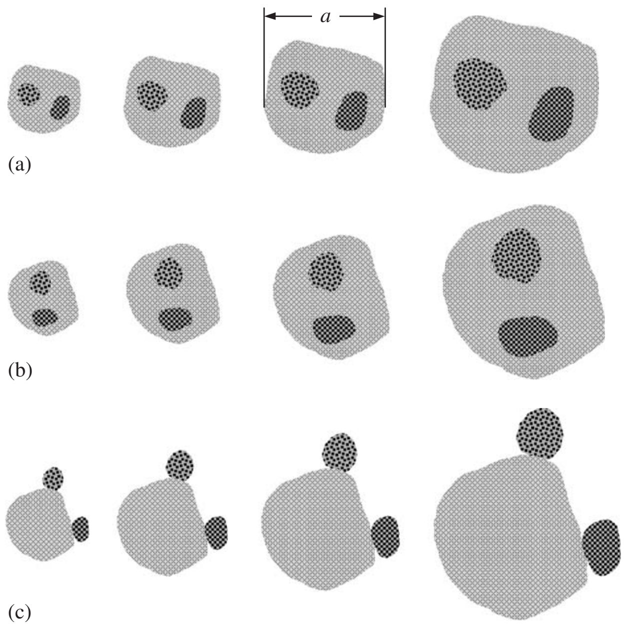

Fig. 2. Three classes of electromagnetically similar objects. Note that the objects in a class have geometrically similar shapes and morphologies as well as identical orientations with respect to the laboratory reference frame.

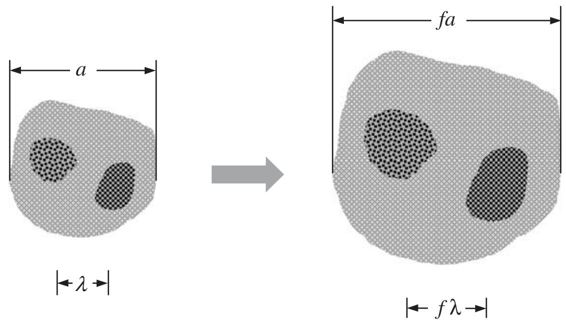

Fig. 3. Alternative formulation of the scale invariance rule.

the same factor f (see Fig. 3) does not change the dimensionless scattering dyadic and the dimensionless amplitude scattering matrix.

The SIR can be very helpful in practice because it makes a single computation or measurement applicable to all couplets (size, wavelength) having the same ratio of size to wavelength, provided that the relative refractive index remains the same. In particular, the scale invariance rule is the basic physical principle of the so-called microwave analog technique. The latter involves measurements of microwave scattering by easily manufactured centimeter-sized objects followed by extrapolation to other wavelengths (e.g., visible or infrared) by keeping the ratio of size to wavelength fixed (e.g., [2] and Section 8.2 of [1]).

Other useful scale-invariant quantities are the ratios $\overleftrightarrow{A}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}})/a$ and $\mathbf{S}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}})/a$. Indeed, since assuming $k_1a = \text{constant}$ yields $k_1\overleftrightarrow{A}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}}) = \text{constant}$ and $k_1\mathbf{S}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}}) = \text{constant}$, dividing the latter two equalities by the first equality must also yield constants. Furthermore, it is straightforward to demonstrate that the SIR is also obeyed by the following quantities:

- the products of k_1^2 and the elements of the phase and scattering matrices;
- the products of k_1^2 and the optical cross sections;
- the products of k_1^2 and the extinction matrix elements;
- the efficiency factors;
- the elements of the normalized scattering matrix;
- the coefficients in the expansions of the elements of the normalized scattering matrix in generalized spherical functions [1,3];
- the single-scattering albedo; and
- the asymmetry parameter.

In general, the SIR applies to any dimensionless scattering characteristic.

4. Final remarks

It should be noted that the methodology used in this paper may lack the mathematical rigor that some mathematics-oriented readers would desire. As one of the reviewers of the initial manuscript remarked, there is no detailed mathematical proof of the existence and uniqueness of solution of Eq. (6) and of the convergence of its iterative solution. While there is no doubt that existence, uniqueness, and convergence are important mathematical issues, it is also obvious that limiting physics research only to problems for which the existence, uniqueness, and convergence of solution have been established mathematically would have a devastating practical effect. Therefore, it appears to be reasonable to claim that the foundation of this paper is sufficiently sound, at least from the standpoint of the physicist.

The same reviewer also suggested that the SIR might be derived directly from the Helmholtz equation supplemented by appropriate boundary conditions [4,5] or from the principle of electrodynamic similitude [6]. In this regard, it should be noted that the approach adopted in this paper is based on the Lippmann–Schwinger equation and totally bypasses the use of the electric field with its "inconvenient" dimension $\mathrm{V\,m^{-1}}$; the only quantities involved have dimensions of the type m^n, where n is an integer. This makes it quite easy to switch to the dimensionless quantities defined by Eqs. (1)–(5) and (8) and prove the SIR in a few simple steps. It is not quite obvious that the same can be done in a straightforward way using the differential equation formalism, especially when the scatterer is an arbitrary heterogeneous body. The SIR does not involve the electric field per say, which makes Eqs. (2.18) and (2.27) of [1] a perfect starting point. It is not clear whether "eliminating" the electric field can be done as easily in the framework of the Helmholtz equation and boundary conditions. Demonstrating that this is possible would be an important result.

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